Testing for Convergence or Divergence of a Series

Many of the series you come across will fall into one of several basic types. Recognizing these types will help you decide which tests or strategies will be most useful in finding whether a series is convergent or divergent.

p-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is…

- convergent if $p > 1$
- divergent if $p \le 1$

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1}$ is…

- convergent if $|r| < 1$
- divergent if $|r| \ge 1$

If a_n has a form that is similar to one of the above, see whether you can use the *comparison test*:

Comparison Test

(Warning! This only works if a_n and b_n are always positive.)

(i) If $a_n \le b_n$ for all *n*, and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $a_n \ge b_n$ for all *n*, and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

Pick $b_n = \frac{1}{n^2}$ (p-series)

$a_n = \frac{1}{n^2 + n} \le \frac{1}{n^2}$, and

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by

(i), $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$ converges.

Example: $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Pick $b_n = \frac{1}{2^n}$ (geometric)

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n - 1} \frac{2^n}{1}$

$= \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, so

$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

Consider a series $\sum b_n$ so that the ratio a_n / b_n cancels the dominant terms in the numerator and denominator of a_n, as in the example to the left. If you know whether $\sum b_n$ converges or not, try using the limit comparison test.

Limit Comparison Test

(Warning! This only works if a_n and b_n are always positive.)

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ (and *c* is finite), then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Some series will "obviously" not converge—recognizing these can save you a lot of time and guesswork.

Test for Divergence

If $\lim_{n\to\infty} a_n \ne 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 - 1}{n^2 + n} = \lim_{n\to\infty} \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n}} = 1 \ne 0$$

so $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n}$ is divergent.

Testing for Convergence or Divergence of a Series (continued)

If a_n can be written as a function with a "nice" integral, the integral test may prove useful:

Integral Test
If $f(n) = a_n$ for all n and $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$, then:

If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.

If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Example: $\sum_{n=1}^\infty \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1,\infty)$.

$\int_1^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}dx$

$= \lim_{t\to\infty}\tan^{-1}x\Big]_1^t = \lim_{t\to\infty}\tan^{-1}t - \frac{\pi}{4}$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, so $\sum_{n=1}^\infty \frac{1}{n^2+1}$ is convergent.

Example: $\sum_{n=1}^\infty (-1)^{n-1}\frac{n^2}{n^3+1}$

(i) $\frac{1}{b_n} = n + \frac{1}{n^2}$, so $\frac{1}{b_{n+1}} = n+1+\frac{1}{(n+1)^2} > n+1$

$\geq n + \frac{1}{n^2} = \frac{1}{b_n}$, so $\frac{1}{b_{n+1}} \geq \frac{1}{b_n}$, so $b_{n+1} \leq b_n$

(ii) $\lim_{n\to\infty}\frac{n^2}{n^3+1} = \lim_{n\to\infty}\frac{1}{n+1/n^2} = 0$

So $\sum_{n=1}^\infty (-1)^{n-1}\frac{n^2}{n^3+1}$ is convergent.

Alternating Series Test
If (i) $b_{n+1} \leq b_n$ for all n and (ii) $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^\infty (-1)^{n-1}b_n$ is convergent.

The following 2 tests prove convergence, but also prove the stronger fact that $\sum |a_n|$ converges (**absolute convergence**).

Ratio Test

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum a_n$ is divergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, use another test.

Example: $\sum_{n=1}^\infty e^{-n}n!$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{-n-1}(n+1)!}{e^{-n}n!}\right|$

$= e^{-1}\lim_{n\to\infty}|n+1| = \infty$, so

$\sum_{n=1}^\infty e^{-n}n!$ is divergent.

Example: $\sum_{n=1}^\infty \frac{n^n}{3^{1+3n}}$

$\lim_{n\to\infty}\left|\sqrt[n]{\frac{n^n}{3^{1+3n}}}\right| = \lim_{n\to\infty}\frac{n}{3^{1/n}3^3}$

$= \frac{1}{27}\lim_{n\to\infty}\frac{n}{3^{1/n}} = \infty$

So $\sum_{n=1}^\infty \frac{n^n}{3^{1+3n}}$ is divergent.

When a_n contains factorials and/or powers of constants, as in the above example, the ratio test is often useful.

Root Test

If $\lim_{n\to\infty}\left|\sqrt[n]{a_n}\right| < 1$, then $\sum a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\left|\sqrt[n]{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum a_n$ is divergent.

If $\lim_{n\to\infty}\left|\sqrt[n]{a_n}\right| = 1$, use another test.